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Stability switch and Hopf bifurcation for a diffusive prey–predator system with delay ☆

Yanbin Tang *, Li Zhou

Department of Mathematics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, PR China

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Abstract

The increasing time delay usually destabilizes any dynamical system. In this paper we give an example that in some special cases the opposite effect can be experienced if the time delay is sufficiently great. We investigate the effect of both the parameter in the time delay kernel and diffusion coefficient on the stability of the positive steady state for a diffusive prey–predator system with delay. We obtain the condition of the occurrence of the stability switches of the positive steady state.

Keywords: Hopf bifurcation; Diffusion; Prey–predator; Time delay; Asymptotically stable; Stability switch

1. Introduction

The prey–predator systems with time delay are deeply concerned by many ecologists and mathematicians [1–16]. Models incorporating time delays in diverse spatially-homogeneous biological systems are extensively reviewed by MacDonald [5] and in the context of predator–prey models by Cushing [1]. Stépán [7], Zhou and Song [10,11] considered the effect of weight function on the stability of the positive steady state. Beretta and Tang [14] discussed the geometric stability switch criterion. With the help of the diffusion coefficients, Zhou and Song [10,11] also

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* Corresponding author.

E-mail address: tangyb@public.wh.hb.cn (Y.B. Tang).

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gave some conditions of the occurrence of the stability switches. In this paper we consider the general two species prey–predator model with diffusion when a distributed delay is introduced:

$$\begin{aligned}
&N_t = dN_{xx} + \varepsilon N(1 - N/k - \alpha P/\varepsilon), \quad 0 < x < \pi,\ t \in R,\\
&P_t = dP_{xx} - \gamma P + \beta P \int_0^\infty N(x, t-\tau) w(\tau)\, d\tau,\\
&N(0,t) = N(\pi,t) = \gamma/\beta, \qquad P(0,t) = P(\pi,t) = \varepsilon(k\beta - \gamma)/(\alpha k \beta),
\end{aligned} \tag{1.1}$$

where $N(x,t)$ and $P(x,t)$ are the population densities of the prey and predator respectively, the parameters $d, k, \alpha, \beta, \gamma$ and ε are positive constants, d is the migration rate, k is the carrying capacity of the prey, α is the rate of predation per predator, β is the rate of conversion of prey into predator, γ is the specific mortality of predator in absence of prey, ε is specific growth rate of prey at zero density in absence of predators. The time delay in prey density in the predator equation could represent effects such as reproduction or maturation time. Here $w(\tau)$ is the delay kernel or memory function. With the help of the scalar weight function $w(\tau)$, the model takes into account the density of the prey in the past, where $\int_0^\infty w(\tau)\, d\tau = 1$. In this paper the memory function $w(\tau) = p\delta(0) + (1-p)\delta(r)$, $0 < p < 1$, with δ being the Dirac delta function, one gets the case of the so-called discrete (or zero) delay. Since we just consider the stability of the positive equilibrium and the Hopf bifurcation, so the initial values are of no importance.

Let us take the transformation $u = N - \gamma/\beta$, $v = P - \varepsilon(k\beta - \gamma)/(\alpha k \beta)$. Similar to [7,10], for the sake of simplicity, let the parameters be $\alpha = \beta = \gamma = \varepsilon = 1$, $k = 2$, then we have

$$\begin{aligned}
&u_t = du_{xx} - u/2 - v - u^2/2 - uv,\\
&v_t = dv_{xx} + \big[pu + (1-p)u(x,t-r)\big]/2 + v\big[pu + (1-p)u(x,t-r)\big],\\
&u(0,t) = u(\pi,t) = v(0,t) = v(\pi,t) = 0.
\end{aligned} \tag{1.2}$$

For the sake of convenience, after a suitable rescaling of the time variable $t = r\tilde{t}$, and then we omit the ˜, we may replace the system (1.2) by the following system:

$$\begin{aligned}
&u_t = r\,du_{xx} - ru/2 - rv - ru^2/2 - ruv,\\
&v_t = r\,dv_{xx} + r\big[pu + (1-p)u(x,t-1)\big]/2 + rv\big[pu + (1-p)u(x,t-1)\big],\\
&u(0,t) = u(\pi,t) = v(0,t) = v(\pi,t) = 0.
\end{aligned} \tag{1.3}$$

In the equations of (1.3) we always agree on that the function valuation will be at (x,t) when we indicate the variables.

The increasing time delay usually destabilizes any dynamical system. In this paper we give an example that in some special cases the opposite effect can be experienced if the time delay is sufficiently great. We investigate the effect of both the parameter in the time delay kernel and diffusion coefficient on the stability of the positive steady state for a diffusive prey–predator system with delay. We obtain the condition of the occurrence of the stability switches of the positive steady state. Regarding the time delay r as the bifurcation parameter, we reduce the original system on the center manifold and calculate the Hopf bifurcation solutions through an iterative process. Finally, we discuss the stability of the Hopf bifurcation solutions.

This paper will be organized as follows. In Section 2, we rewrite the system (1.3) into an abstract operator differential equation. In Section 3, we discuss the stability of the trivial solution of (1.3) and give the condition of the occurrence of the stability switches of the positive steady state. In Section 4, we use the center manifold theorem to reduce the system and investigate the stability of the Hopf bifurcation solutions. In Section 5, we give a conclusion.

2. Abstract operator differential equation

For the sake of convenience, similar to the method described in [2,7–11], we first transform the problem (1.3) into an abstract operator differential equation. Let $X = L^p(0,\pi) \times L^p(0,\pi)$, with the norm $\|(f_1, f_2)^T\|_X = \sum_{i=1}^{2} \|f_i\|_p$ or other equivalent norms are equipped on X. Let $\mathcal{B} = \mathcal{C}([-1,0], X)$, with the norm $\|\cdot\|_{\mathcal{B}} = \sup_{[-1,0]} \|\cdot\|_X$.

The nonlinear operator $F : \mathcal{D}(F) \to \mathcal{B}$ is defined as follows:

$$F(v) = \begin{cases} \partial v/\partial\theta, & \theta \in [-1,0), \\ \begin{pmatrix} r(dv_{1xx} - \frac{1}{2}v_1(x,t) - v_2(x,t) - \frac{1}{2}v_1^2(x,t) - v_1v_2) \\ r(dv_{2xx} + (\frac{1}{2} + v_2(x,t))(pv_1(x,t) + (1-p)v_1(x,t-1))) \end{pmatrix}, & \theta = 0, \end{cases}$$

for $v = (v_1(x,t+\theta), v_2(x,t+\theta))^T \in \mathcal{D}(F)$, besides smoothness we restrict $v_i|_{x=0} = v_i|_{x=\pi} = 0$, $i = 1,2$. Suppose that $(u(x,t+\theta), v(x,t+\theta))^T$ is a solution of (1.3) on $[0,\pi] \times [-1,T]$, we denote $U(t) = (u(x,t+\theta), v(x,t+\theta))^T$, $-1 \leqslant \theta \leqslant 0$, then $U(t) \in \mathcal{B}$ is well defined on $[0,T]$ and satisfies

$$dU/dt = F(U). \tag{2.1}$$

Conversely, the solutions of the problem (2.1) are also the solutions of the problem (1.3), so we will directly discuss the corresponding abstract operator differential equation (2.1) below.

3. The stability analysis of the trivial solution

Linearizing Eq. (2.1) at the trivial solution $U = (0,0)^T$, we obtain a linear system

$$dv/dt = Av, \quad v \in \mathcal{D}(A), \tag{3.1}$$

where

$$Av = \begin{cases} \partial v/\partial\theta, & \theta \in [-1,0), \\ \begin{pmatrix} r(dv_{1xx} - \frac{1}{2}v_1(x,t) - v_2(x,t)) \\ r(dv_{2xx} + \frac{1}{2}(pv_1(x,t) + (1-p)v_1(x,t-1))) \end{pmatrix}, & \theta = 0, \end{cases}$$

and $v(x,\theta)(t) = (v_1(x,t+\theta), v_2(x,t+\theta))^T \in \mathcal{D}(A)$, $\mathcal{D}(A) = \{v(x,\theta)(t) \in \mathcal{B} \mid v(x=0,\theta)(t) = v(x=\pi,\theta)(t) = 0,\ \forall t \geqslant 0\}$. Now we investigate the eigenvalue problem of the operator A,

$$A\phi = \lambda\phi, \quad \phi(x,\theta) \in \mathcal{D}(A) \subset \mathcal{B}, \tag{3.2}$$

that is,

$$\left(\frac{\partial\phi_1}{\partial\theta}, \frac{\partial\phi_2}{\partial\theta}\right)^T = \lambda(\phi_1,\phi_2)^T, \quad -1 \leqslant \theta < 0, \tag{3.3}$$

$$\begin{aligned} &\frac{\partial\phi_1}{\partial\theta}(x,0) = r\,d\phi_{1xx}(x,0) - \frac{r}{2}\phi_1(x,0) - r\phi_2(x,0), \quad \theta = 0, \\ &\frac{\partial\phi_2}{\partial\theta}(x,0) = r\,d\phi_{2xx}(x,0) + \frac{r}{2}\big(p\phi_1(x,0) + (1-p)\phi_1(x,-1)\big), \quad \theta = 0. \end{aligned} \tag{3.4}$$

We obtain from (3.3), $\phi(x,\theta) = e^{\lambda\theta}(\phi_1^*(x), \phi_2^*(x))^T$, substitute it into (3.4), we get

$$
\begin{aligned}
\lambda\phi_1^*(x) &= r\,d\phi_{1xx}^*(x) - \frac{r}{2}\phi_1^* - r\phi_2^*,\\
\lambda\phi_2^*(x) &= r\,d\phi_{2xx}^*(x) + \frac{r}{2}\big[p + (1-p)e^{-\lambda}\big]\phi_1^*,
\end{aligned}
\tag{3.5}
$$

since $\phi(x,\theta) \in \mathcal{D}(A)$, so $\phi_i^*(0) = \phi_i^*(\pi) = 0$, $i = 1, 2$. Because of the right-hand side of (3.5) just contains the unknown functions and their second-order derivatives, hence we may choose $(\phi_1^*(x), \phi_2^*(x))^T = (\phi_1^n(x), \phi_2^n(x))^T \sin nx$, $n = 1, 2, \ldots$, substitute it into (3.5), we get

$$
\lambda\begin{pmatrix}\phi_1^n(x)\\ \phi_2^n(x)\end{pmatrix} = \begin{pmatrix} -rn^2d - \frac{r}{2} & -r\\ \frac{r}{2}[p + (1-p)e^{-\lambda}] & -rn^2d\end{pmatrix}\begin{pmatrix}\phi_1^n(x)\\ \phi_2^n(x)\end{pmatrix}
\tag{3.6}
$$

therefore we obtain the characteristic equations which assume the form

$$
D_n(\lambda, r) = \lambda^2 + \left(\frac{1}{2} + 2n^2d\right)r\lambda + r^2\left[\frac{1}{2}\big(p + (1-p)e^{-\lambda}\big) + n^2d\left(\frac{1}{2} + n^2d\right)\right] = 0.
$$

For the stability of the trivial solution of (2.1), we need to determine the distribution of the characteristic roots. Since each root of $D_n(\lambda, r) = 0$, $n = 1, 2, \ldots$, is a function of the parameters p, r, d, n and when $r = 0$ all the roots of the characteristic equation lie in the left-hand half plane (we have the stability of the trivial solution for zero delay), $\lambda = 0$ is not a characteristic root, a necessary condition for a stability switch is that the characteristic equation has a pair of pure imaginary roots $\lambda = \pm iy$, that is, a pair of conjugate complex roots cross the imaginary axis. We substitute $\lambda = iy$, $y > 0$ in $D_n(\lambda, r) = 0$, then we get

Theorem 3.1. *For any $d > 0$, $0 < p < 1$, the trivial solution of* (2.1) *is stable if the time delay $0 < r < (1 + 4d)/(1 - p)$.*

Proof. Let $D_n(iy, r) = M_n(y, r) + iS_n(y, r)$, where

$$
\begin{aligned}
M_n(y, r) &= -y^2 + r^2\big[p + (1-p)\cos y + n^2d\big(1 + 2n^2d\big)\big]/2,\\
S_n(y, r) &= \big[\big(1 + 4n^2d\big)yr - (1-p)r^2\sin y\big]/2,
\end{aligned}
$$

and $y > 0$. It is clear that $S_n(k\pi, r) > 0$, $k = 1, 2, \ldots$, this means that $y = k\pi$, $k = 1, 2, \ldots$, are not the roots of $D_n(iy, r) = 0$. Now for $y > 0$ and $y \neq k\pi$ we have

$$
S_n(y, r) \geqslant \left(\frac{1}{2} + 2n^2d\right)yr - \frac{1-p}{2}r^2|\sin y| = \frac{1-p}{2}r|\sin y|\left(\frac{1 + 4n^2d}{1-p}\frac{y}{|\sin y|} - r\right),
$$

since $\inf_{(0,+\infty)} \frac{y}{|\sin y|} = 1$, so $S_n(y, r) > 0$ if $r < \frac{1+4d}{1-p}$ and $y > 0$. This implies that $D_n(\lambda, r) = 0$ has not any pure imaginary roots $\lambda = iy$ if $r < \frac{1+4d}{1-p}$, and this excludes the possibility of changes of the stability. Therefore, the trivial solution of (2.1) is locally asymptotic stable if $0 < r < \frac{1+4d}{1-p}$ due to its stability at $r = 0$. This completes the proof. □

Remark. For any $d > 0$ and $0 < p < 1$, all roots of $D_n(\lambda, r) = 0$ are in the left-hand half plane if $0 < r < (1 + 4d)/(1 - p)$.

Now we want to seek the conditions of the occurrence of the pure imaginary roots. Separating the real and imaginary parts of $D_n(iy, r) = 0$, we have

$$M_n(y,r) = -y^2 + r^2\big[p + (1-p)\cos y + n^2 d\big(1 + 2n^2 d\big)\big]/2 = 0, \tag{3.7}$$

$$S_n(y,r) = \big[\big(1 + 4n^2 d\big)yr - (1-p)r^2 \sin y\big]/2 = 0, \tag{3.8}$$

from (3.8), we get

$$y\big(1 + 4n^2 d\big) = r(1-p)\sin y. \tag{3.9}$$

Since the left-hand side of (3.9) is positive, so $\sin y > 0$. From (3.7) we have

$$2y^2 = r^2\big[\big(1 + 2n^2 d\big)n^2 d + p + (1-p)\cos y\big], \tag{3.10}$$

so from (3.9), (3.10) we get

$$\cos^2 y + \frac{2(\frac{1}{2} + 2n^2 d)^2}{1-p}\cos y + \frac{2p + 2n^2 d(1 + 2n^2 d)}{(1-p)^2}\left(\frac{1}{2} + 2n^2 d\right)^2 - 1 = 0. \tag{3.11}$$

Notice that if $y > 0$ is the positive root of (3.11), then $\pm iy$ are the pure imaginary roots of $D_n(\lambda, r) = 0$, that is, $D_n(\pm iy, r) = 0$. Denote $z = \cos y$ and let $n = 1$, we get

$$z^2 + \frac{2(\frac{1}{2} + 2d)^2}{1-p}z + \frac{2p + 2d(1 + 2d)}{(1-p)^2}\left(\frac{1}{2} + 2d\right)^2 - 1 = 0. \tag{3.12}$$

The discriminant of the quadratic equation (3.12) is

$$\begin{aligned}\Delta(d,p) &= \frac{4}{(1-p)^2}\left[\left(\frac{1}{2} + 2d\right)^4 - 2\big(p + d(1 + 2d)\big)\left(\frac{1}{2} + 2d\right)^2 + (1-p)^2\right] \\ &= \frac{4}{(1-p)^2}\Delta_1(d,p),\end{aligned}$$

where

$$\Delta_1(d,p) = p^2 - \big(8d^2 + 4d + 5/2\big)p + d^2 + d/2 + 17/16. \tag{3.13}$$

It is easy to check that $\Delta_1(d,0) = (d + 1/4)^2 + 1 > 0$, $\Delta_1(d,1) = -7(d + 1/4)^2 < 0$, and

$$\frac{\partial \Delta_1(d,p)}{\partial p} = -2(1-p) - 2\left(\frac{1}{2} + 2d\right)^2 < 0, \quad 0 < p < 1.$$

Therefore $\Delta_1(d,p) = 0$ defines uniquely a function $p = p(d)$ in the strip for $d \geqslant 0$, $0 < p < 1$, such that $\Delta(d, p(d)) = 0$. In fact,

$$p = p(d) = \frac{1}{2}\left[8d^2 + 4d + \frac{5}{2} - \sqrt{\left(8d^2 + 4d + \frac{5}{2}\right)^2 - 4\left(d^2 + \frac{d}{2} + \frac{17}{16}\right)}\right]. \tag{3.14}$$

Lemma 3.1. *The function $p(d)$ is monotone decreasing and satisfies that*

$$p(0) = 5/4 - \sqrt{1/2}, \qquad \lim_{d \to +\infty} p(d) = 1/8.$$

Proof. By direct computation we get

$$p'(d) = (2 + 8d)\left(1 - \frac{4d^2 + 2d + \frac{9}{8}}{\sqrt{(4d^2 + 2d + \frac{5}{4})^2 - (d^2 + \frac{d}{2} + \frac{17}{16})}}\right) < 0.$$

This completes the proof. $\square$

Denote the region $\mathcal{D} = \{(d, p) \mid d \geqslant 0,\ \frac{1}{8} < p < 1\}$.

Lemma 3.2. *When $(d, p) \in \mathcal{D}$ such that $p > p(d)$, then the equations $D_n(\lambda, r) = 0$, $n = 1, 2, \ldots$, do not have any pure imaginary roots.*

Proof. It is clear that $p > p(d) > p(n^2 d)$, $n = 2, 3, \ldots$, so we have $\Delta(n^2 d, p) < 0$, $n = 1, 2, \ldots$, Eq. (3.11) has not any positive roots, it means that $D_n(\lambda, r) = 0$ has not any pure imaginary roots. This completes the proof. □

Let the curve AQ represent the graph of $p = p(d)$ (see Fig. 1), when (d, p) is a point below the curve AQ, then $p < p(d)$, that is, $\Delta(d, p) > 0$, therefore Eq. (3.12) has two real roots $z_2(d, p) < z_1(d, p) \leqslant 0$ for $(d, p) \in \mathcal{D}$ and $p < p(d)$ such that

$$z_{1,2}(d, p) = \frac{1}{1-p}\left[-\left(\frac{1}{2} + 2d\right)^2 \pm \sqrt{\Delta_1(d, p)}\right]. \tag{3.15}$$

In the region $\mathcal{D}$, we can find a parabola $p = p^*(d) = 1 - (\frac{1}{2} + 2d)^2$, it intersects the curve $p = p(d)$ at $D(\frac{\sqrt{21}-3}{12}, \frac{5}{12})$.

In the region $\mathcal{D}_1 = \{(d, p) \in \mathcal{D} \mid p^*(d) < p < p(d)\}$ there is a parabola

$$p = p_+(d) = \left[9 - (1 + 4d)^2\right]/16, \quad (d, p) \in \mathcal{D}_1, \tag{3.16}$$

such that $z_1(d, p_+(d)) = -1$. Similarly, in the region $\mathcal{D}_2 = \{(d, p) \in \mathcal{D} \mid p < p^*(d)$ and $p < p(d)\}$, we can find a parabola

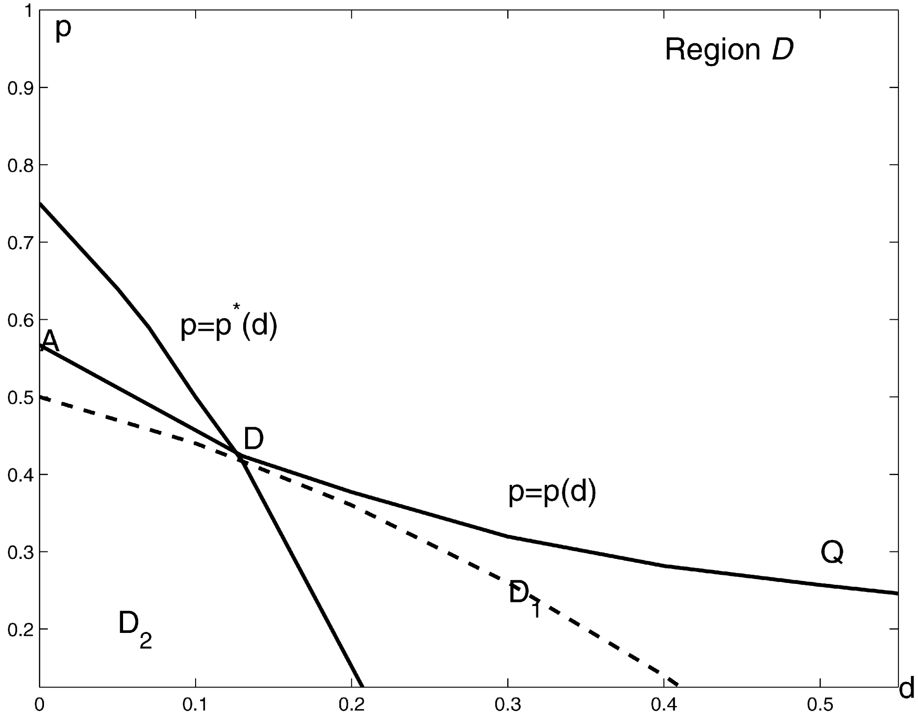

Fig. 1. The curves of the functions $p = p(d)$, $p = p^*(d)$.

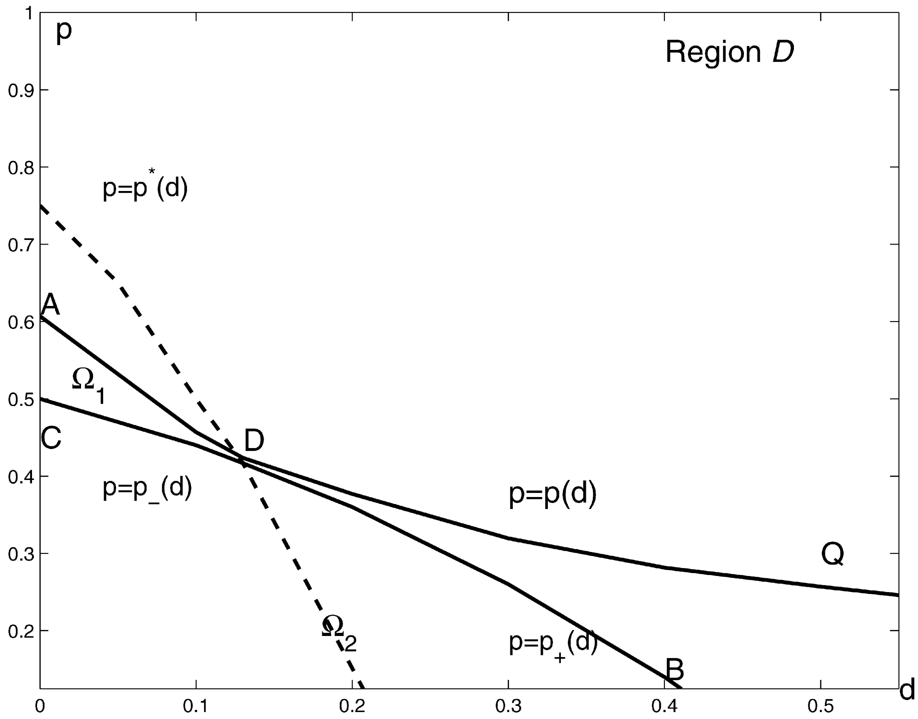

Fig. 2. The curves of the functions $p = p(d)$, $p = p^*(d)$, $p = p_+(d)$, $p = p_-(d)$.

$$p = p_-(d) = \left[9 - (1 + 4d)^2\right]/16, \quad (d, p) \in \mathcal{D}_2, \tag{3.17}$$

such that $z_2(d, p_-(d)) = -1$.

It is very interesting that both $p_+(d)$ and $p_-(d)$ have the same formula but they are defined in different regions of (d, p), respectively. By direct computation, we know that the curves $p = p_+(d)$, $p = p_-(d)$, $p = p(d)$ and $p = p^*(d)$ intersect at point $D(\frac{\sqrt{21}-3}{12}, \frac{5}{12})$ (see Fig. 2).

Denote Ω_1 the region $\widehat{ADCA}$ bounded by the curves $\widehat{AD}$, $\widehat{DC}$ and segment $\overline{AC}$, and Ω_2 bounded by the curves $\widehat{CD}$, $\widehat{DB}$ and segments $\overline{BO}$, $\overline{OC}$ (see Fig. 2), then we have

Theorem 3.2. *The trivial solution of* (2.1) *is stable for any* $r \geqslant 0$ *if* $(d, p) \in \mathcal{D} \setminus \{\Omega_1 \cup \Omega_2\}$.

Proof. If $(d, p) \in \mathcal{D}$, $p > p(d)$, then $\Delta_1(d, p) < 0$, Lemma 3.2 implies that $D_n(\lambda, r) = 0$, $n = 1, 2, \ldots,$ do not have any pure imaginary roots. If $(d, p) \in \mathcal{D} \setminus \{\Omega_1 \cup \Omega_2\}$ and $p < p(d)$, as $z_1(d, p) \leqslant 0$ and $\frac{\partial \Delta_1(d,p)}{\partial p} < 0$, we have

$$\begin{aligned}
\frac{\partial z_1(d, p)}{\partial p} &= \frac{1}{(1-p)^2}\left[-\left(\frac{1}{2} + 2d\right)^2 + \sqrt{\Delta_1(d, p)}\right] + \frac{1}{2(1-p)\sqrt{\Delta_1(d, p)}} \frac{\partial \Delta_1(d, p)}{\partial p} \\
&= \frac{1}{1-p}\left[z_1(d, p) + \frac{1}{2\sqrt{\Delta_1(d, p)}} \frac{\partial \Delta_1(d, p)}{\partial p}\right] \\
&= \frac{(\frac{1}{2} + 2d)^2}{(1-p)^2\sqrt{\Delta_1(d, p)}}\left[-\sqrt{\Delta_1(d, p)} - p - \frac{3}{4}\right] < 0,
\end{aligned} \tag{3.18}$$

hence for $p > p_+(d)$, we have $z_1(d,p) < z_1(d,p_+(d)) = -1$, that is, $z_2(d,p) < z_1(d,p) < -1$. Then we have $\cos y = z_i < -1$, $i = 1,2$. It is impossible. Therefore $D_1(\lambda, r) = 0$ do not have any pure imaginary roots $\pm iy$, so the eigenvalues are all in the left half plane, this implies that the trivial solution is stable. This completes the proof. $\square$

When $(d,p) \in \mathcal{D}_2$, we have

$$\begin{aligned}
\frac{\partial z_2(d,p)}{\partial p} &= \frac{1}{(1-p)^2}\left[-\left(\frac{1}{2}+2d\right)^2 - \sqrt{\Delta_1(d,p)}\right] - \frac{1}{2(1-p)\sqrt{\Delta_1(d,p)}}\frac{\partial \Delta_1(d,p)}{\partial p} \\
&= \frac{1}{(1-p)^2\sqrt{\Delta_1}}\left[-\left(\frac{1}{2}+2d\right)^2\sqrt{\Delta_1} - \Delta_1 - \frac{1-p}{2}\frac{\partial \Delta_1}{\partial p}\right] \\
&= \frac{1}{(1-p)^2\sqrt{\Delta_1}}\left[-\left(\frac{1}{2}+2d\right)^2\sqrt{\Delta_1} - \Delta_1 + (1-p)^2 + (1-p)\left(\frac{1}{2}+2d\right)^2\right] \\
&= \frac{(\frac{1}{2}+2d)^2}{(1-p)^2\sqrt{\Delta_1(d,p)}}\left[-\sqrt{\Delta_1(d,p)} + p + \frac{3}{4}\right] \\
&= \frac{(\frac{1}{2}+2d)^2}{(1-p)^2\sqrt{\Delta_1(d,p)}}\frac{(2d^2+d+1)(8p-1)}{2(\sqrt{\Delta_1(d,p)}+p+\frac{3}{4})} > 0.
\end{aligned} \tag{3.19}$$

For $(d,p) \in \Omega_1$, we have $p > p_-(d)$, therefore, (3.19) implies that $z_2(d,p) > z_2(d,p_-(d)) = -1$. Since $\frac{\partial z_1(d,p)}{\partial p} < 0$, $\frac{\partial \Delta_1(d,p)}{\partial p} < 0$ and $p_-(d) < p < p(d)$ for $(d,p) \in \Omega_1$, we have

$$\Delta_1(d,p) < \Delta_1\bigl(d,p_-(d)\bigr) = \left[1 - \bigl(9-(1+4d)^2\bigr)/16 - \left(\frac{1}{2}+2d\right)^2\right]^2,$$

thus,

$$\begin{aligned}
z_1(d,p) &= \frac{1}{1-p}\left[\sqrt{\Delta_1(d,p)} - \left(\frac{1}{2}+2d\right)^2\right] < \frac{1}{1-p}\left[\sqrt{\Delta_1\bigl(d,p_-(d)\bigr)} - \left(\frac{1}{2}+2d\right)^2\right] \\
&= \frac{1}{1-p}\left[\frac{7}{16} - \frac{7}{4}\left(\frac{1}{2}+2d\right)^2\right] < 0,
\end{aligned}$$

therefore we have

$$-1 < z_2(d,p) < z_1(d,p) < 0, \quad (d,p) \in \Omega_1. \tag{3.20}$$

Lemma 3.3. *When $(d,p) \in \Omega_1$, there exist two sequences $\{r_k\}$ and $\{\bar{r}_k\}$ such that the equations $D_1(\lambda, r_k) = 0$ and $D_1(\lambda, \bar{r}_k) = 0$ have pure imaginary roots $\pm iy_k$ and $\pm i\bar{y}_k$, respectively. Moreover*

(1) *The sum of multiplicities of the characteristic roots with positive real parts is increasing by two when r passes increasingly through the values r_k.*
(2) *The sum of multiplicities of the characteristic roots with positive real parts is decreasing by two when r passes increasingly through the values $\bar{r}_k$.*

Proof. Denote

$$y_k = \arccos z_1(d,p) + 2(k-1)\pi, \quad w_k = -\arccos z_1(d,p) + 2k\pi, \quad k = 1,2,\ldots,$$

since $\frac{\pi}{2} \leqslant \arccos z_1(d,p) \leqslant \pi$, so $\sin w_k < 0$, we discard w_k. By means of (3.9), we get

$$r_k = \frac{(1+4d)y_k}{(1-p)\sin y_1}, \quad y_1 = \arccos z_1(d,p), \quad y_k = y_1 + 2(k-1)\pi, \quad k = 1,2,\ldots. \tag{3.21}$$

Obviously $D_1(\pm i y_k, r_k) = 0$. Similarly, we have

$$\bar{r}_k = \frac{(1+4d)\bar{y}_k}{(1-p)\sin \bar{y}_1}, \quad \bar{y}_1 = \arccos z_2(d,p), \quad \bar{y}_k = \bar{y}_1 + 2(k-1)\pi, \quad k = 1,2,\ldots. \tag{3.22}$$

In order to prove the first part of the theorem we consider the implicit function $\lambda = \lambda(r)$ which is defined by the equation $D_1(\lambda, r) = 0$ in the neighborhood of (iy_k, r_k). We know that the conclusion (1) holds if

$$\operatorname{sgn}\left\{\operatorname{Re}\left(\frac{d\lambda}{dr}\bigg|_{(iy_k,r_k)}\right)\right\} = 1. \tag{3.23}$$

In fact,

$$\frac{d\lambda}{dr} = -\frac{(\frac{1}{2}+2d)\lambda + r[p+(1-p)e^{-\lambda}] + 2rd(\frac{1}{2}+d)}{2\lambda + (\frac{1}{2}+2d)r - \frac{r^2}{2}(1-p)e^{-\lambda}}. \tag{3.24}$$

Using (3.9), (3.10), we get

$$\begin{aligned}
\operatorname{sgn}\left\{\operatorname{Re}\left(\frac{d\lambda}{dr}\bigg|_{(\lambda=iy_k,\, r=r_k)}\right)\right\} &= \operatorname{sgn}\left\{\left[y^2 r(1-p)\cos y + y^2 r\left(\frac{1}{2}+2d\right)^2\right]\bigg|_{(y=y_k,\, r=r_k)}\right\} \\
&= \operatorname{sgn}\left[(1-p)z_1(d,p) + \left(\frac{1}{2}+2d\right)^2\right] \\
&= \operatorname{sgn}\sqrt{\Delta_1(d,p)} = 1.
\end{aligned}$$

Similarly, the conclusion (2) holds since

$$\begin{aligned}
\operatorname{sgn}\left\{\operatorname{Re}\left(\frac{d\lambda}{dr}\bigg|_{(\lambda=i\bar{y}_k,\, r=\bar{r}_k)}\right)\right\} &= \operatorname{sgn}\left[(1-p)z_2(d,p) + \left(\frac{1}{2}+2d\right)^2\right] \\
&= \operatorname{sgn}\left(-\sqrt{\Delta_1(d,p)}\right) = -1.
\end{aligned}$$

This completes the proof. □

Remark. When $(d,p) \in \Omega_1$ such that $(n^2 d, p) \in \Omega_1$, then there exist two sequences $\{r_k^n\}$ and $\{\bar{r}_k^n\}$ such that the equations $D_n(\lambda, r_k^n) = 0$ and $D_n(\lambda, \bar{r}_k^n) = 0$ have pure imaginary roots $\pm i y_k^n$ and $\pm i \bar{y}_k^n$, respectively.

Corollary 3.1. *The trivial solution of* (2.1) *is stable if* $(d,p) \in \Omega_1$ *and* $r < r_1(d,p)$.

By the proof of Lemma 3.3, we can show that $\{r_k\}$ and $\{\bar{r}_k\}$ are equal difference sequences $r_k = r_1 + (k-1)D, \bar{r}_k = \bar{r}_1 + (k-1)\overline{D}$, where

$$D = \frac{2\pi(1+4d)}{(1-p)\sin y_1}, \quad \overline{D} = \frac{2\pi(1+4d)}{(1-p)\sin \bar{y}_1}, \quad D < \overline{D}. \tag{3.25}$$

Theorem 3.3. *For any $(d, p) \in \Omega_1$, suppose that $r_2(d, p) < \bar{r}_1(d, p)$, then the trivial solution of* (2.1) *is unstable if $r > r_1(d, p)$.*

Proof. Since $r_2(d, p) < \bar{r}_1(d, p)$, there are at least two complex roots in the right half plane as $r_1(d, p) < r \leqslant \bar{r}_1(d, p)$, hence the trivial solution is unstable in this case. By means of $r_2(d, p) < \bar{r}_1(d, p)$, and $D < \overline{D}$, the amount of complex roots of $D_1(\lambda, r) = 0$ in the right half plane is $2([\frac{r-r_1}{D}] - [\frac{r-\bar{r}_1}{\overline{D}}]) \geqslant 2$ as $r > \bar{r}_1(d, p)$, where $[a]$ denotes the integer part of a. Therefore the trivial solution is also unstable if $r > \bar{r}_1(d, p)$. This completes the proof. $\square$

Lemma 3.4. *Suppose that $0 < d < d_0$, then there exists a curve $p = p_*(d)$ such that $r_2(d, p) < \bar{r}_1(d, p)$ as $p > p_*(d)$ and $r_2(d, p) > \bar{r}_1(d, p)$ as $p < p_*(d)$.*

Proof. Set

$$g(d, p) = r_2(d, p) - \bar{r}_1(d, p) = \frac{(y_1 + 2\pi)(1 + 4d)}{(1 - p)\sin y_1} - \frac{\bar{y}_1(1 + 4d)}{(1 - p)\sin \bar{y}_1}, \tag{3.26}$$

As $(d, p) \in \Omega_1$, from (3.20), $-1 < z_2(d, p) < z_1(d, p) < 0$. Since $\frac{\partial z_1}{\partial p} < 0$ (see (3.18)) and $\frac{\partial z_2}{\partial p} > 0$ (see (3.19)), for $y_1 = \arccos z_1(d, p)$ and $\bar{y}_1 = \arccos z_2(d, p)$, we have $\frac{\partial y_1}{\partial p} > 0$ and $\frac{\partial \bar{y}_1}{\partial p} < 0$. Now we consider the function $g(d, p)$,

$$\begin{aligned} \frac{\partial g}{\partial p}(d, p) &= \frac{\partial y_1}{\partial p}\frac{1 + 4d}{(1 - p)\sin y_1} + (y_1 + 2\pi)\frac{\partial}{\partial p}\left(\frac{1 + 4d}{(1 - p)\sin y_1}\right) \\ &\quad - \frac{\partial \bar{y}_1}{\partial p}\frac{1 + 4d}{(1 - p)\sin \bar{y}_1} - \bar{y}_1\frac{\partial}{\partial p}\left(\frac{1 + 4d}{(1 - p)\sin \bar{y}_1}\right) = I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.27}$$

Since $\frac{\partial y_1}{\partial p} > 0$ and $\frac{\partial \bar{y}_1}{\partial p} < 0$, so $I_1 > 0$, $I_3 > 0$. By means of $-1 < z_1(d, p) < 0$ and $\frac{\pi}{2} < y_1 < \pi$, we have

$$\frac{\partial}{\partial p}\left(\frac{1 + 4d}{(1 - p)\sin y_1}\right) = -\frac{1 + 4d}{(1 - p)^2 \sin^2 y_1}\left[-\sin y_1 + (1 - p)\cos y_1 \frac{\partial y_1}{\partial p}\right] > 0,$$

so $I_2 > 0$. From $z_2(d, p) = \cos \bar{y}_1 = \frac{1}{1-p}[-(\frac{1}{2} + 2d)^2 - \sqrt{\Delta_1(d, p)}]$, we have

$$\sin^2 \bar{y}_1(d, p) = \frac{(2p - \frac{1}{4} - 2\sqrt{\Delta_1(d, p)} - (\frac{1}{2} + 2d)^2)(\frac{1}{2} + 2d)^2}{(1 - p)^2}. \tag{3.28}$$

Differentiate (3.28) with respect to p, we get

$$2\sin \bar{y}_1 \frac{\partial \sin \bar{y}_1(d, p)}{\partial p} = \frac{2((\frac{1}{2} + 2d)^2 + \sqrt{\Delta_1})(\frac{3}{4} + p - \sqrt{\Delta_1})(\frac{1}{2} + 2d)^2}{(1 - p)^3\sqrt{\Delta_1}}, \tag{3.29}$$

but

$$\frac{\partial}{\partial p}\left(\frac{1}{(1 - p)\sin \bar{y}_1}\right) = \frac{(\sin \bar{y}_1)^2 - (1 - p)\sin \bar{y}_1 \frac{\partial}{\partial p}\sin \bar{y}_1}{(1 - p)^2(\sin \bar{y}_1)^3}. \tag{3.30}$$

Using (3.28), (3.29), we get

$$\begin{aligned}\operatorname{sgn} I_4 &= \operatorname{sgn}\left[(\sin \bar{y}_1)^2 - (1-p)\sin \bar{y}_1 \frac{\partial}{\partial p}\sin \bar{y}_1\right] \\ &= \operatorname{sgn}\left[\Delta_1 + (1-p)\sqrt{\Delta_1} + \left(\frac{3}{4}+p\right)\left(\frac{1}{2}+2d\right)^2\right] = 1,\end{aligned}$$

hence $I_4 > 0$. Therefore we obtain $\frac{\partial g}{\partial p}(d,p) > 0$ for $(d,p) \in \Omega_1$. Since

$$g\big(d, p_-(d)\big) = -\infty, \quad g\big(d, p(d)\big) = \frac{2\pi(1+4d)}{(1-p)\sin \bar{y}_1(d, p(d))} > 0,$$

thus there exists a unique function $p = p_*(d)$ such that $g(d, p_*(d)) = 0$. This completes the proof. □

Using the above lemmas and corollaries we get

Theorem 3.4. *Suppose that $(d,p) \in \Omega_1$, $d > \frac{1}{4}d_0$, $p > p_*(d)$, then there exists $r_c(d,p)$ such that*:

(1) *When r increases from 0 to $r_c(d,p)$, the stability switches of zero solution of* (2.1) *from stability to instability to stability occur, and zero solution is unstable if $r > r_c(d,p)$.*
(2) *The Hopf bifurcation will occur at every critical point at which the stability of trivial solution would change.*

Proof. Refer to the proof of Theorem 6 in [10]. □

Remark. If $(d,p) \in \{(d,p) \in \Omega_1 \mid d < \frac{1}{4}d_0\}$, the effect of r on the stability of the trivial solution of (2.1) is determined by discussing the distribution of roots of $D_n(\lambda, r) = 0, n = 2, 3, \ldots.$

Theorem 3.5. *Suppose that $(d,p) \in \Omega_2$, then the trivial solution of* (2.1) *is stable for $r < r_1(d,p)$ and unstable for $r > r_1(d,p)$.*

Proof. When $(d,p) \in \Omega_2$, $z_2(d,p) < -1$, i.e., $\cos \bar{y}_k < -1$, the sequence $(\bar{y}_k, \bar{r}_k)$ does not exist. According to the proof of Lemma 3.3, we know that the trivial solution of (2.1) is stable for $0 \leqslant r < r_1(d,p)$ and unstable for $r > r_1(d,p)$. This completes the proof. □

Sum up the above theorems, we indicate the effect of two parameters on the stability of trivial solution of (2.1).

4. Center manifold and Hopf bifurcation solution

By means of the above theorems, there are many critical points if we choose r as bifurcation parameter. In the following we will have a great eyes to $r_1(d,p)$ as representation, we would state the computation of the center manifold and the reduction of the Hopf bifurcation solution. Hereafter, we denote $r = r_1(d,p)$, $y = y_1(d,p)$. First we would compute the eigenfunctions belonging to the eigenvalue iy of the operator A. Let

$$\phi(x,\theta) = e^{iy\theta}(\alpha_1 + i\beta_1, \alpha_2 + i\beta_2)^T \sin x,$$

substitute it into (3.6) we get

$$\begin{pmatrix} -r(d+\frac{1}{2}) & -r & y & 0 \\ y & 0 & r(d+\frac{1}{2}) & r \\ \frac{r}{2}(p+(1-p)\cos y) & -rd & \frac{r}{2}(1-p)\sin y & y \\ \frac{r}{2}(1-p)\sin y & y & -\frac{r}{2}(p+(1-p)\cos y) & rd \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix} = 0, \tag{4.1}$$

we can prove that the rank of coefficient matrix of above equations is 2, choose $\beta_1 = 1, \alpha_1 = 0$, so $\alpha_2 = y/r$, $\beta_2 = -(1+2d)/2$, we get

$$\phi(x,\theta) = \left[\begin{pmatrix} 0 \\ \frac{y}{r} \end{pmatrix} \cos y\theta + \begin{pmatrix} -1 \\ \frac{1}{2}+d \end{pmatrix} \sin y\theta + i \begin{pmatrix} 1 \\ \frac{1}{2}+d \end{pmatrix} \cos y\theta + i \begin{pmatrix} 0 \\ \frac{y}{r} \end{pmatrix} \sin y\theta \right] \sin x. \tag{4.2}$$

In order to decompose the basic space we have to compute the eigenfunction belonging to the eigenvalue $-iy$ of the conjugate operator A^*, where

$$A^*\psi = \begin{cases} -\frac{\partial\psi}{\partial\sigma}, & \sigma \in (0,1], \\ \begin{pmatrix} r(d\psi_{1xx} - \frac{1}{2}\psi_1(x,0) + \frac{1}{2}(p\psi_2(x,0) + (1-p)\psi_2(x,1))) \\ r(d\psi_{2xx} - r\psi_1(x,0)) \end{pmatrix}, & \sigma = 0. \end{cases} \tag{4.3}$$

Suppose that the eigenfunction belonging to $-iy$ assumes the form

$$\psi(x,\sigma) = e^{iy\sigma}(a_1 + ib_1, a_2 + ib_2)^T \sin x, \quad (x,\sigma) \in [0,\pi] \times [0,1]. \tag{4.4}$$

Similarly, we get $a_1 = -da_2 - yb_2/r$, $b_1 = ya_2/r - db_2$, where a_2, b_2 will be determined by the normalized conditions

$$\langle n_1, s_1 \rangle = 1, \qquad \langle n_1, s_2 \rangle = 0, \tag{4.5}$$

where $\phi = s_1 + is_2$, $\psi = n_1 + in_2$ and $\langle \cdot,\cdot \rangle$ is the generalized scalar product (Hermite) which is defined as follows

$$\langle \psi, \phi \rangle = \frac{2}{\pi} \int_0^{\pi} \left[\psi^T(x,0)\phi(x,0) + \int_{-1}^{0} \psi^T(x,\xi+1) \begin{pmatrix} 0 & 0 \\ \frac{r(1-p)}{2} & 0 \end{pmatrix} \phi(x,\xi)\, d\xi \right] dx.$$

It is clear that $\langle \psi, A\phi \rangle = \langle A^*\psi, \phi \rangle$. From (4.5) we get the equations of a_2, b_2,

$$\frac{y}{r}a_2 + \frac{r(1-p)}{2} \int_{-1}^{0} \big[a_2 \cos y(\xi+1) - b_2 \sin y(\xi+1)\big](-\sin y\xi)\, d\xi = 1,$$

$$-\left(\frac{1}{2}+d\right)a_2 - \frac{y}{r}b_2 + \frac{r(1-p)}{2} \int_{-1}^{0} \big[a_2 \cos y(\xi+1) - b_2 \sin y(\xi+1)\big] \cos y\xi\, d\xi = 0,$$

so we have

$$a_2 = \frac{-\frac{y}{r} - \frac{r(1-p)}{2}k_3}{(\frac{r(1-p)}{2}k_2 - \frac{y}{r})(\frac{y}{r} + \frac{r(1-p)}{2}k_3) + \frac{r(1-p)}{2}k_4(\frac{1}{2} + 2d - \frac{r(1-p)}{2}k_1)},$$

$$b_2 = \frac{\frac{1}{2} + 2d - \frac{r(1-p)}{2}k_1}{(\frac{r(1-p)}{2}k_2 - \frac{y}{r})(\frac{y}{r} + \frac{r(1-p)}{2}k_3) + \frac{r(1-p)}{2}k_4(\frac{1}{2} + 2d\frac{r(1-p)}{2}k_1)},$$

where

$$k_1 = \cos y/2 + \sin y/(2y), \qquad k_2 = -\sin y/2, \qquad k_3 = -k_2,$$
$$k_4 = \cos y/2 - \sin y/(2y).$$

Let us consider the transformation

$$v_1 = \langle n_1, U\rangle, \qquad v_2 = \langle n_2, U\rangle, \qquad w = U - v_1 s_1 - v_2 s_2, \tag{4.6}$$

where $v_1, v_2 : [0, +\infty) \to \mathcal{R}$, $w : [0, +\infty) \to \mathcal{B}$. It means that the time dependent scalars v_1 and v_2 are the coordinates of U in the directions of s_1 and s_2 respectively while the other part of U are contained in the infinite dimensional w. Now let us see the new form of (2.1)

$$\begin{aligned}
\frac{dv_1}{dt} &= yv_2 - \frac{2}{\pi}\int_0^\pi n_1^T(x,0)\begin{pmatrix} \frac{r}{2}U_1^2(x,0) + rU_1(x,0)U_2(x,0) \\ -\frac{r}{2}U_2(x,0)(pU_1(x,0) + (1-p)U_1(x,-1)) \end{pmatrix} dx, \\
\frac{dv_2}{dt} &= -yv_1 - \frac{2}{\pi}\int_0^\pi n_2^T(x,0)\begin{pmatrix} \frac{r}{2}U_1^2(x,0) + rU_1(x,0)U_2(x,0) \\ -\frac{r}{2}U_2(x,0)(pU_1(x,0) + (1-p)U_1(x,-1)) \end{pmatrix} dx, \\
\frac{dw}{dt} &= Aw + \frac{2}{\pi}\int_0^\pi n_1^T(x,0)\begin{pmatrix} \frac{r}{2}U_1^2(x,0) + rU_1(x,0)U_2(x,0) \\ -\frac{r}{2}U_2(x,0)(pU_1(x,0) + (1-p)U_1(x,-1)) \end{pmatrix} dx\, s_1 \\
&\quad + \frac{2}{\pi}\int_0^\pi n_2^T(x,0)\begin{pmatrix} \frac{r}{2}U_1^2(x,0) + rU_1(x,0)U_2(x,0) \\ -\frac{r}{2}U_2(x,0)(pU_1(x,0) + (1-p)U_1(x,-1)) \end{pmatrix} dx\, s_2 \\
&\quad - \begin{cases} 0, & \theta \in [-1,0), \\ \begin{pmatrix} \frac{r}{2}U_1^2(x,0) + rU_1(x,0)U_2(x,0) \\ -\frac{r}{2}U_2(x,0)(pU_1(x,0) + (1-p)U_1(x,-1)) \end{pmatrix}, & \theta = 0. \end{cases}
\end{aligned} \tag{4.7}$$

Suppose that w is the restriction of U on the center manifold, then by means of the tangent relation between the center manifold and the span subspace of s_1 and s_2, we have

$$w(x,\theta,t) = \frac{1}{2}\big(h_{11}(x,\theta)v_1^2 + 2h_{12}(x,\theta)v_1v_2 + h_{22}(x,\theta)v_2^2\big) + \cdots, \tag{4.8}$$

then the system (4.7) satisfies the following form:

$$\begin{aligned}
\frac{dv_1}{dt} &= yv_2 + M_2(v_1,v_2) + M_3\big(v_1, v_2, w_1(0,t), w_1(-1,t), w_2(0,t)\big) + \text{h.o.t.}, \\
\frac{dv_2}{dt} &= -yv_1 + N_2(v_1,v_2) + N_3\big(v_1, v_2, w_1(0,t), w_1(-1,t), w_2(0,t)\big) + \text{h.o.t.}, \\
\frac{dw}{dt} &= Aw - M_2 s_1 - N_2 s_2 + \begin{cases} 0, & \theta \in [-1,0), \\ \overline{w}(v_1,v_2), & \theta = 0, \end{cases} + \text{h.o.t.},
\end{aligned} \tag{4.9}$$

where h.o.t. represents the higher-order terms, and

$$
\begin{aligned}
M_2 = \frac{8}{3\pi}\bigg[& a_2(1-p)y\sin y v_1^2 + \bigg(dya_2 + \frac{y^2}{r}b_2 + \bigg(y\big(p+(1-p)\cos y\big) \\
& - r\left(\frac{1}{2}+d\right)(1-p)\sin y\bigg)a_2\bigg)v_1v_2 \\
& - \left(d^2ra_2 + dyb_2 + r\left(\frac{1}{2}+d\right)\big(p+(1-p)\cos y\big)a_2\right)v_2^2\bigg],
\end{aligned}
$$

$$
\begin{aligned}
N_2 = \frac{8}{3\pi}\bigg[& b_2(1-p)y\sin y v_1^2 + \bigg(-\frac{y^2}{r}a_2 + \bigg(dy + y\big(p+(1-p)\cos y\big) \\
& - r\left(\frac{1}{2}+d\right)(1-p)\sin y\bigg)b_2\bigg)v_1v_2 \\
& + \left(dya_2 - \left(d^2r + r\left(\frac{1}{2}+d\right)\big(p+(1-p)\cos y\big)\right)b_2\right)v_2^2\bigg],
\end{aligned}
$$

$$
\begin{aligned}
M_3 = \frac{2}{\pi}\int_0^\pi \bigg\{&\left[\frac{y^2}{r}b_2 + (d+p)ya_2\right]v_1\widehat{w}_1(x,0,t) \\
& + \left[\left(\frac{1}{2}-d\right)yb_2 + \left(d\left(\frac{1}{2}-d\right) - \left(\frac{1}{2}+d\right)p\right)ra_2\right]v_2\widehat{w}_1(x,0,t) \\
& + \big((1-p)a_2r\sin y\big)v_1\widehat{w}_2(x,0,t) \\
& + \big[yb_2 + \big(d+p+(1-p)\cos y\big)ra_2\big]v_2\widehat{w}_2(x,0,t) \\
& + (1-p)ya_2v_1\widehat{w}_1(x,-1,t) - \left(\frac{1}{2}+d\right)(1-p)ra_2v_2\widehat{w}_1(x,-1,t)\bigg\}\sin^2 x\,dx,
\end{aligned}
$$

$$
\begin{aligned}
N_3 = \frac{2}{\pi}\int_0^\pi \bigg\{&\left[-\frac{y^2}{r}a_2 + (d+p)yb_2\right]v_1\widehat{w}_1(x,0,t) \\
& + \left[-\left(\frac{1}{2}-d\right)ya_2 + \left(d\left(\frac{1}{2}-d\right) - \left(\frac{1}{2}+d\right)p\right)rb_2\right]v_2\widehat{w}_1(x,0,t) \\
& + \big((1-p)b_2r\sin y\big)v_1\widehat{w}_2(x,0,t) \\
& + \big[-ya_2 + \big(d+p+(1-p)\cos y\big)rb_2\big]v_2\widehat{w}_2(x,0,t) \\
& + (1-p)yb_2v_1\widehat{w}_1(x,-1,t) - \left(\frac{1}{2}+d\right)(1-p)rb_2v_2\widehat{w}_1(x,-1,t)\bigg\}\sin^2 x\,dx,
\end{aligned}
$$

$$
\overline{w} = \sin^2 x \begin{pmatrix} -yv_1v_2 + drv_2^2 \\ (1-p)y\sin y v_1^2 + L \end{pmatrix},
$$

$$
L = \left(y\big(p+(1-p)\cos y\big) - \left(\frac{1}{2}+d\right)(1-p)\sin y\right)v_1v_2 - \left(\frac{1}{2}+d\right)(1-p)\cos y v_2^2,
$$

$$
\widehat{w} = \begin{pmatrix} \widehat{w}_1 \\ \widehat{w}_2 \end{pmatrix} = \frac{1}{2}\big(h_{11}(x,\theta)v_1^2 + 2h_{12}(x,\theta)v_1v_2 + h_{22}(x,\theta)v_2^2\big).
$$

Differentiate (4.8) with respect to t, then we get

$$
dw/dt = -yh_{12}v_1^2 + (h_{11} - h_{12})yv_1v_2 + yh_{22}v_2^2 + \text{h.o.t.} \tag{4.10}
$$

On the other hand, substitute (4.8) into the third equation of (4.9), we get

$$\frac{dw}{dt} = \begin{cases} \frac{1}{2}\frac{\partial}{\partial\theta}(h_{11}(x,\theta)v_1^2 + 2h_{12}(x,\theta)v_1v_2 + h_{22}(x,\theta)v_2^2), & \theta \in [-1,0), \\ \begin{pmatrix} rd\widehat{w}_{1xx}(x,0,t) - \frac{r}{2}\widehat{w}_1(x,0,t) - r\widehat{w}_2(x,0,t) \\ rd\widehat{w}_{2xx}(x,0,t) + \frac{r}{2}(p\widehat{w}_1(x,0,t) + (1-p)\widehat{w}_1(x,-1,t)) \end{pmatrix} + \overline{w}, & \theta = 0, \end{cases}$$
$$- M_2 s_1 - N_2 s_2 + \text{h.o.t.} \tag{4.11}$$

Now equating the coefficients of v_1^2, v_1v_2, v_2^2 of (4.10) and (4.11) respectively, then when $\theta \in [-1,0)$, we get

$$\frac{\partial}{\partial\theta}\begin{pmatrix} h_{11} \\ h_{12} \\ h_{22} \end{pmatrix} = \begin{pmatrix} 0 & 2yI & 0 \\ yI & 0 & -yI \\ 0 & 2yI & 0 \end{pmatrix}\begin{pmatrix} h_{11} \\ h_{12} \\ h_{22} \end{pmatrix} + (l\cos y\theta + m\sin y\theta)\sin x, \tag{4.12}$$

where I is a 2×2 unit matrix, h_{ij} are two-dimensional vectors, l, m are 6-dimensional vectors which are determined by $M_2s_1 + N_2s_2$. The general solution of (4.12) is

$$\begin{aligned}(h_{11}, h_{12}, h_{22})^T &= e_1r_1(x) + e_2r_2(x) + e_{31}\cos 2y\theta r_3(x) + e_{32}\sin 2y\theta r_3(x) \\ &\quad + e_{41}\sin 2y\theta r_4(x) + e_{42}\cos 2y\theta r_4(x) + e_{51}\cos 2y\theta r_5(x) \\ &\quad + e_{52}\sin 2y\theta r_5(x) + e_{61}\sin 2y\theta r_6(x) + e_{62}\cos 2y\theta r_6(x) \\ &\quad + (\bar{h}_{11}, \bar{h}_{12}, \bar{h}_{22})^T,\end{aligned} \tag{4.13}$$

where

$$e_1 = \begin{pmatrix} 1\\0\\0\\0\\1\\0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0\\1\\0\\0\\0\\1 \end{pmatrix}, \quad e_{31} = \begin{pmatrix} 1\\0\\0\\0\\-1\\0 \end{pmatrix}, \quad e_{32} = \begin{pmatrix} 0\\0\\1\\0\\0\\0 \end{pmatrix}, \quad e_{41} = \begin{pmatrix} 1\\0\\0\\0\\-1\\0 \end{pmatrix},$$

$$e_{42} = \begin{pmatrix} 0\\0\\-1\\0\\0\\0 \end{pmatrix}, \quad e_{51} = \begin{pmatrix} 0\\1\\0\\0\\0\\-1 \end{pmatrix}, \quad e_{52} = \begin{pmatrix} 0\\0\\0\\1\\0\\0 \end{pmatrix}, \quad e_{61} = \begin{pmatrix} 0\\1\\0\\0\\0\\-1 \end{pmatrix}, \quad e_{62} = \begin{pmatrix} 0\\0\\0\\-1\\0\\0 \end{pmatrix},$$

and $(\bar{h}_{11}, \bar{h}_{12}, \bar{h}_{22})^T$ is a particular solution. According to Eqs. (4.10), (4.11) at $\theta = 0$ and using (4.13), we get the boundary value problems of differential equations on $r_i(x)$, $i = 1, 2, \ldots, 6$, we know that each $r_i(x)$, $i = 1, 2, \ldots, 6$, has the expression $\sum c_k^{(i)} \sin kx$. In order to determine the coefficient $c_k^{(i)}$, $i = 1, 2, \ldots, 6$, we will expand the nonhomogeneous terms in series form, then we transform the boundary value problem of the differential equations into the linear algebraic equations, the details are omitted here (refer to [11]).

Substitute these $r_i(x)$ into (4.13), the functions $h_{ij}(x)$ are obtained, then we get $\widehat{w}_i(x,\theta,t)$ from (4.8) and the expressions of M_3, N_3. Finally, by (4.9) we finish the reduction of the original system on the center manifold, the first two equations in (4.9) have the form

$$
\begin{aligned}
dv_1/dt &= yv_2 + a_{20}v_1^2 + a_{11}v_1v_2 + a_{02}v_2^2 + b_{30}v_1^3 + b_{21}v_1^2v_2 + b_{12}v_1v_2^2 + b_{03}v_2^3 + \text{h.o.t.},\\
dv_2/dt &= -yv_1 + c_{20}v_1^2 + c_{11}v_1v_2 + c_{02}v_2^2 + d_{30}v_1^3 + d_{21}v_1^2v_2 + d_{12}v_1v_2^2 + d_{03}v_2^3\\
&\quad + \text{h.o.t.},
\end{aligned}
$$

by means of the method described in [4], we transform these equations into the Poincaré normal form: $dz/dt = -iyz + C_0|z|^2z + \text{h.o.t.}$, where

$$
\begin{aligned}
\operatorname{Re} C_0 &= \big[(a_{20} + c_{11} - a_{02})(c_{02} + c_{20}) + (c_{02} - a_{11} + c_{20})(a_{02} + a_{20})\big]\big/(8y)\\
&\quad + 3(b_{30} + d_{30})/8 + (b_{12} + d_{21})/8.
\end{aligned}
$$

To sum up, we give the following result by the Hopf bifurcation theorem.

Theorem 4.1. *If* $\operatorname{Re} C_0 < 0$, *then the problem* (1.1) *exhibits the supercritical periodic bifurcation solution at* $r = r_{d,p}$, *it is stable and has the following asymptotic expressions*:

$$
\begin{pmatrix} N(x,t)\\ P(x,t)\end{pmatrix} = \begin{pmatrix} 1\\ \frac{1}{2}\end{pmatrix} + A_1\sqrt{r - r_1}\left(\begin{pmatrix} 0\\ \frac{y_1}{r_1}\end{pmatrix}\cos\frac{y_1}{r_1}t + \begin{pmatrix} 1\\ -\frac{1}{2} - d\end{pmatrix}\sin\frac{y_1}{r_1}t\right)\sin x + \text{h.o.t.},
$$

where

$$
\begin{aligned}
A_1^2 = &\left[r_1^2\big(p + (1-p)\cos y_1 + d(2d+1)\big)\left(\left(2d + \frac{1}{2}\right) - \frac{r_1}{2}(1-p)\cos y_1\right)\right.\\
&\left. - \left(\left(\frac{1}{2} + 2d\right)y_1 - (1-p)r_1\sin y_1\right)\left(2y_1 + \frac{1-p}{2}r_1^2\sin y_1\right)\right]\\
&\times\left[-\operatorname{Re} C_1(0)\left[\left(\left(2d + \frac{1}{2}\right)r_1 - \frac{1-p}{2}r_1\cos y_1\right)^2\right.\right.\\
&\left.\left. + \left(2y_1 + \frac{1-p}{2}r_1\sin y_1\right)^2\right]\right]^{-1}.
\end{aligned}
$$

The Hopf bifurcation will occur at every critical point at which the stability of the trivial solution would change, the system possess a certain "oscillatory" behavior (Theorem 4.1). If $\operatorname{Re} C_0 < 0$, the supercritical Hopf bifurcation solution is stable. We try to confirm the supercritical Hopf bifurcation by numerical simulation, but the condition and the formulas are very difficult to verify in practice, we will consider this for further details.

5. Conclusion

A diffusive prey–predator system which describes a predator–prey interaction subject to delay effects is considered. A rather complete picture is drawn of certain qualitative aspects of the positive steady state as it is a function of the parameter p in the time delay kernel, diffusion coefficient d and the time delay r in the system. If the time delay is smaller than a critical value, $r \in (0, \frac{1+4d}{1-p})$, for any $d > 0$, $0 < p < 1$, the steady state of (1.3) is stable (Theorem 3.1). When $(d, p) \in \mathcal{D} \setminus \{\Omega_1 \cup \Omega_2\}$, for any time delay $r > 0$, the steady state is also stable (Theorem 3.2), it means that such delay cannot alter the qualitative behavior of the steady state. When $(d, p) \in \Omega_1$, the system exhibits very complicated phenomena. For example (Theorem 3.4), under certain conditions, there exists a critical value $r_c(d, p)$ of the time delay, when r increases from 0 to $r_c(d, p)$, the stability switches of zero solution of (2.1) from stability to instability to stability

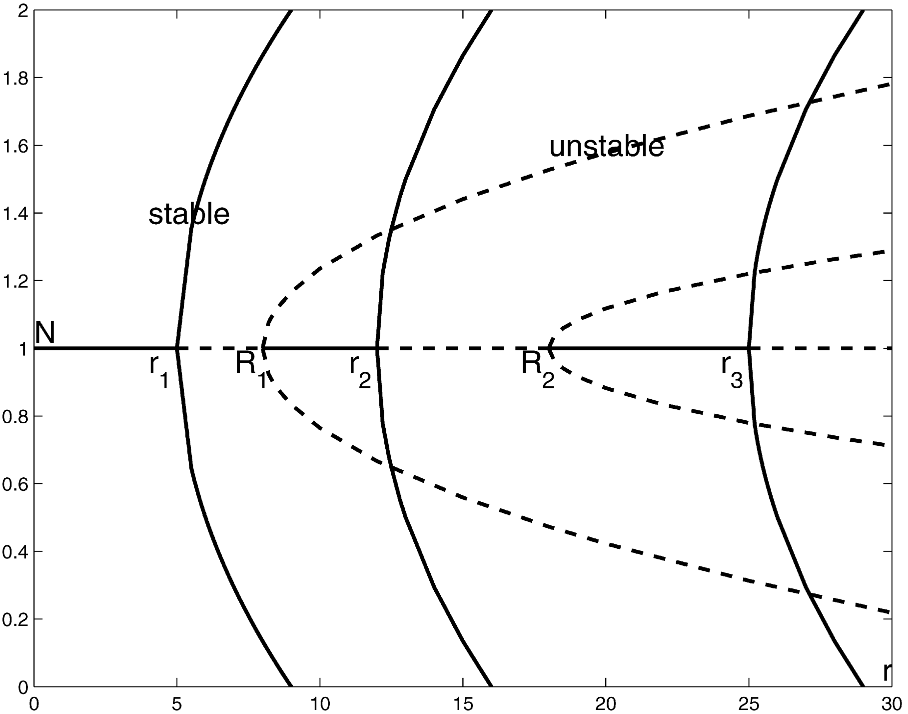

Fig. 3. Bifurcation diagram of Theorem 3.4.

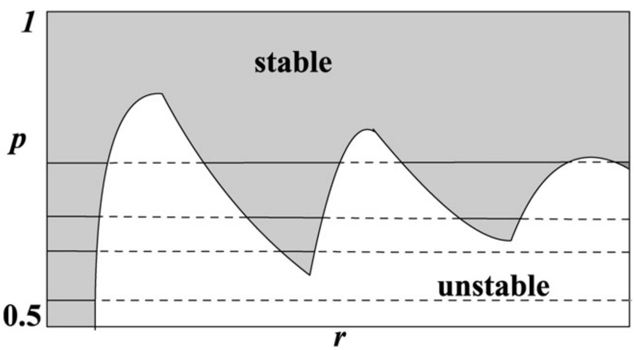

Fig. 4. Stability chart for the fixed diffusion coefficient $d \in (d_0/4, d_0)$.

occur (see Fig. 3), the stability of zero solution can change a finite number of times as r is increased and zero solution eventually becomes unstable for sufficiently large value $r > r_c(d, p)$. For the fixed diffusion coefficient $d \in (\frac{1}{4}d_0, d_0)$, we can get the stability chart in Fig. 4, it is very similar to the Stépán [7] for the absence of the diffusion coefficient.

We expect that stability result of the positive steady state holds true for the parameter $p \in (0, 1)$, similar to the Stépán [7] with the absence of the diffusion coefficient, but here we give the stability result in Theorem 3.4 only for $p \in (\frac{1}{8}, 1)$. In fact, for $p \in (0, \frac{1}{8})$, the stability switch does not occur.

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